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One-dimensional $SU(3)$ bosons with δ-function interaction

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Abstract

In this paper, we solve one-dimensional $SU(3)$ bosons with repulsive δ-function interaction using the Bethe ansatz method. The features of the ground and low-lying excited states are studied by both numerical and analytical methods. We show that the ground state is a $SU(3)$ colour ferromagnetic state. The configurations of quantum numbers for the ground state are given explicitly. For a finite N system, the spectra of low-lying excitations and the dispersion relations of four possible elementary particles (holon, antiholon, σ-coloron and ω-coloron) are obtained by solving the Bethe ansatz equation numerically. The thermodynamic equilibrium of the system at finite temperature is studied by using the strategy of the thermodynamic Bethe ansatz; we give a revised Gaudin–Takahashi equation which is useful for the numerical method. Thermodynamic quantities, such as specific heat, are obtained for some special cases. We find that the magnetic property of the model in the high-temperature regime is dominated by Curie's law, $\chi \propto 1/T$, and the system has Fermi-liquid-like specific heat in the strong coupling limit at low temperature.

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1. Introduction

One of the main goals of theoretical physics during the past 40 years has been to understand quantum systems involving many particles. Once the interaction between these particles is taken into account, the problem becomes complicated. Meanwhile, as long as their interaction is not sufficiently weak, the perturbative methods that were powerful in many quantum mechanics text books become unreliable. In one dimension, various non-perturbative methods have been proposed, among which the impact of exactly solvable theoretical models is undeniable. Particles with δ-function interaction provide a simple but interesting model. Lieb and Liniger [1] first solved a Bose system under the periodic boundary condition in the case of spin-0 or in the absence of internal degrees of freedom. The method they used is

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nowadays referred to as the coordinate Bethe ansatz. The extension of the periodic boundary condition to the boundary condition of the potential well of infinite depth was made by Gaudin [2] and Woynarivich [3]. The first attempt to develop the method applied in [1] to deal with spin-$1/2$ fermions was made by McGuire [4] who can, however, deal with the case of only one spin-down and keeping other spins up. A further step of considering two spin-down and other spins up was made by Flicker and Lieb [5]. Gaudin [6] and Yang [7] successfully solved the problem for an arbitrary number of spins in the state of spin-down. In Yang's paper [7], the first non-trivial case of the Yang–Baxter equation was introduced. Actually, the strategy for general multi-component systems was proposed in [7] although the explicit solution was given only for spin-$1/2$ particles (it is then the Fermi system).

The literature in [7] was later extended by Sutherland [8] to any irreducible representation of the permutation group. Actually, both Yang and Sutherland adopted an antisymmetric wavefunction under permutating indistinguishable particles. Thus, Yang solved the problem of two-component fermions and Sutherland solved N-component fermions using the coordinate Bethe ansatz. As the two-component system is mostly associated with the 'spin-$1/2$' system, which is conventionally referred to as the Fermi system, the coordinate Bethe ansatz has not been used for two-component Bose systems until recently [9].

Along with the developments of the quantum inverse scattering method, Kulish [10] discussed the multi-component nonlinear Schrödinger equation in terms of the quantum inverse scattering method (QISM) in order to re-derive the Bethe ansatz equations of Yang [7] and Sutherland [8]. Kulish explicitly formulated the two-component case and conjectured that Sutherland's results would be obtained by repeating his procedure $n-2$ times. Actually, this is not possible because Kulish [10] adopted commutation (instead of anticommutation) relations, but the system both Yang and Sutherland considered is the Fermi system. It is now clear that the first quantization form of the system which Kulish considered ought to be a system of $SU(n)$ bosons with δ-function interaction. The QISM was also employed for the nonlinear Schrödinger equation for a graded matrix but it breaks the Yang–Baxter relation at first [11], which was noticed and overcome later [12].

Although the Bethe ansatz equations for two-component bosons was formulated earlier by Kulish, the nature of the ground state and the properties of low-lying excitations were not exposed until the work of Li *et al* [9]. We know that not only can the two-component Bose gas be formed in magnetically trapped ^{87}Rb [13], but also a three-component Bose gas can be produced in an optically trapped ^{23}Na [14]. It will be valuable to study the model of the three-component Bose system. In this paper, we study a system of three-component bosons with $SU(3)$ symmetry in one dimension. On the basis of the Bethe ansatz equations, we discuss the ground state, the low-lying excited states and the thermodynamics of the system at finite temperature, and we try to obtain thermal coefficients for some special cases. Our paper is organized as follows. In the following section we introduce the model and the corresponding Bethe ansatz equations for charge rapidity and colour rapidities. In section 3, we explicitly show that the ground state is a colour ferromagnetic state and how the quantum numbers in the Bethe ansatz equations should be taken for the ground state. In section 4, we study the low-lying excited states extensively by analysing the possible variations in the sequence of quantum numbers. The numerical results of energy momentum spectra for each excitation are given. Furthermore, the dispersion relations of four possible elementary particles are obtained. In section 5 we discuss the general thermodynamics of the system with the strategy of the thermodynamic Bethe ansatz (TBA), which was proposed by Yang and Yang [19] when they studied Bose gas with δ-function interaction in one dimension. In section 6 we discuss the system for some special cases and we obtain some analytical results. In section 7 we give a brief summary.

2. The model and its Bethe ansatz solution

We consider an interacting $SU(3)$ Bose field in a one-dimensional ring of length L. The model Hamiltonian of the system reads

$$\mathcal{H}_0 = \int \mathrm{d}x \left[\sum_a \partial_x \psi_a^* \partial_x \psi_a + \frac{c}{2} \sum_{a,b} \psi_a^* \psi_a \psi_b^* \psi_b \right] \tag{1}$$

where the natural unit is adopted for simplicity. Here, c is the coupling constant and $a, b = 1, 2, 3$ (which we call colours hereafter) denote the three states that carry out the fundamental representation of the $SU(3)$ group. The fields obey the following commutation relations:

$$[\psi_a^*(x), \psi_b(y)] = \sum_n \delta_{ab} \delta(x - y - nL). \tag{2}$$

In the terminology of group theory, the three states $|1\rangle$, $|2\rangle$ and $|3\rangle$ are labelled by weight vectors $(1/2, 0)$, $(-1/2, 1/2)$ and $(0, -1/2)$, respectively. The two $su(2)$ subalgebra in the $su(3)$ Lie algebra are $[T^+, T^-] = 2T^z$ and $[U^+, U^-] = 2U^z$. With the help of these 'flipping' operators, $U^\pm$ and $T^\pm$, we can generate the three states from the highest weight state $|1\rangle$, i.e.

$$T^-|1\rangle = |2\rangle \qquad U^-|2\rangle = |3\rangle.$$

With additional commutation relations defined by

$$V^+ = [T^+, U^+] \qquad V^- = [T^-, U^-]$$

the Chevalley bases of the $su(3)$ Lie algebra consist of eight generators $\{T^\pm, U^\pm, V^\pm, T^z, U^z\}$.

In the domain with $x_i \neq x_j$, the Hamiltonian (1) reduces to that for free bosons and its eigenfunction is therefore just the superposition of plane waves. When two particles collide with each other, a scattering process occurs. The coordinate Bethe ansatz embodies that this process is purely elastic, i.e. exchange of their momenta. So for a given momentum $k = (k_1, k_2, \ldots, k_N)$, the scattering momenta include all permutations of the components of k. Thus, for the case of N bosons, because the Hamiltonian is invariant under the action of the permutation group S_N, we can adopt the following Bethe ansatz wavefunction

$$\Psi_a(x) = \sum_{P \in S_N} A_a(P, Q)\, \mathrm{e}^{\mathrm{i}(Pk|Qx)} \qquad x \in \mathcal{C}(Q) \tag{3}$$

where $a = (a_1, a_2, \ldots, a_N)$, a_j denotes the colour label of the jth particle, Pk denotes the image of a given $k := (k_1, k_2, \ldots, k_N)$ by a mapping $P \in S_N$, and the coefficients $A(P, Q)$ are functionals of P and Q where Q denotes a permutation of the coordinates which define a region with $0 < x_{Q_1} < x_{Q_2} < \ldots < x_{Q_N} < L$. For a Bose system, the wavefunction is supposed to be symmetric under any permutation of both coordinates and colour indices, i.e.

$$(\Pi^j \Psi)_a(x) = \Psi_a(x) \tag{4}$$

where $\Pi^j : \{a_1, \ldots, a_j, a_{j+1}, \ldots\} \mapsto \{a_1, \ldots, a_{j+1}, a_j, \ldots\}$ and $(\Pi^j \Psi)_a$ is well defined by $\Psi_{\Pi^j a}(\Pi^j x)$. Furthermore, using the identity $(Pk|\Pi^i Qx) = (\Pi^i Pk|Qx)$ and the rearrangement theorem of group theory, we have the following consequence from equation (4):

$$A_a(P, \Pi^i Q) = A_{\Pi^j a}(\Pi^i P, Q). \tag{5}$$

The δ-function term in the Hamiltonian (1) contributes a boundary condition across the hyper-plane $x_{Q_j} = x_{Q_{j+1}}$

$$\begin{aligned} i((Pk)_j - (Pk)_{j+1})&[A_a(P, \Pi^j Q) - A_a(\Pi^j P, \Pi^j Q) - A_a(P, Q) + A_a(\Pi^j P, Q)] \\ &= 2c[A_a(P, Q) + A_a(\Pi^j P, Q)]. \end{aligned} \tag{6}$$

By making use of the relations (5) and (6) together with the continuity condition, we can obtain the following relation

$$A_a(\Pi^j P, Q) = \frac{i[(Pk)_j - (Pk)_{j+1}]\mathcal{P}^j + c}{i[(kP)_j - (Pk)_{j+1}] - c} A_a(P, Q) \tag{7}$$

where $\mathcal{P}^j$ permutates the colour labels of bosons located at x_{Q_j} and $x_{Q_{j+1}}$.

Applying the periodic boundary condition $\Psi\big(\ldots, x_{Q_j}, \ldots\big) = \Psi\big(\ldots, x_{Q_j} + L, \ldots\big)$ and making use of the standard procedure of the QISM [8, 10, 15], we can obtain the Bethe ansatz equations:

$$\begin{aligned}
e^{ik_j L} &= -\prod_{l=1}^{N} \frac{k_j - k_l + ic}{k_j - k_l - ic} \prod_{\nu=1}^{M} \frac{k_j - \lambda_\nu - ic/2}{k_j - \lambda_\nu + ic/2} \\
1 &= -\prod_{l=1}^{N} \frac{\lambda_\gamma - k_l - ic/2}{\lambda_\gamma - k_l + ic/2} \prod_{\nu=1}^{M} \frac{\lambda_\gamma - \lambda_\nu + ic}{\lambda_\gamma - \lambda_\nu - ic} \prod_{\alpha=1}^{M'} \frac{\lambda_\gamma - \mu_\alpha - ic/2}{\lambda_\gamma - \mu_\alpha + ic/2} \\
1 &= -\prod_{\nu=1}^{M} \frac{\mu_\beta - \lambda_\nu - ic/2}{\mu_\beta - \lambda_\nu + ic/2} \prod_{\alpha=1}^{M'} \frac{\mu_\beta - \mu_\alpha + ic}{\mu_\beta - \mu_\alpha - ic}.
\end{aligned} \tag{8}$$

λ and μ are $SU(3)$ colour rapidities. There are $M - M'$ particles in the state $|2\rangle$, M' in $|3\rangle$ and $N - M$ in $|1\rangle$. We would like to mention here that the state obtained above is the highest weight state among the multiplet of $SU(3)$ representation labelled $(N/2 + M'/2 - M, M/2 - M')$. The other states in the multiplets can be generated by iterate application of the flipping operators T^- and U^-.

Taking the logarithm of equations (8) we have secular equations

$$\begin{aligned}
k_j L &= 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l) + \sum_{\nu=1}^{M} \Theta_{-1/2}(k_j - \lambda_\nu) \\
2\pi J_\gamma &= \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\nu=1}^{M} \Theta_1(\lambda_\gamma - \lambda_\nu) + \sum_{\alpha=1}^{M'} \Theta_{-1/2}(\lambda_\gamma - \mu_\alpha) \\
2\pi J'_\beta &= \sum_{\nu=1}^{M} \Theta_{-1/2}(\mu_\beta - \lambda_\nu) + \sum_{\alpha=1}^{M'} \Theta_1(\mu_\beta - \mu_\alpha)
\end{aligned} \tag{9}$$

where $\Theta_n(x) = -2\tan^{-1}(x/nc)$. The quantum number I_j for charge rapidity k_j takes an integer or half-integer depending on whether $N - M$ is odd or even. The quantum number J_γ and J'_β for $SU(3)$ colour rapidities λ_γ and μ_β take an integer or half-integer depending on whether $N - M - M'$ and $M - M'$ are odd or even, respectively. Once all roots $\{k_j, \lambda_\gamma, \mu_\beta\}$ are solved from the above equations (9) for a given set of quantum numbers $\{I_j, J_\gamma, J'_\beta\}$, the energy and momentum will be calculated by

$$E = \sum_{j=1}^{N} k_j^2 \qquad p = \frac{2\pi}{L}\left[\sum_{j=1}^{N} I_j - \sum_{\gamma=1}^{M} J_\gamma - \sum_{\beta=1}^{M'} J'_\beta\right] \tag{10}$$

where the second equation of equation (10) is obtained from equation (9) directly.

For a state with real roots (k, λ, μ), we may define the distribution densities $\rho(k), \sigma(\lambda)$ and $\omega(\mu)$:

$$\begin{aligned}
\rho(k_j) &= 1/L(k_{j+1} - k_j) \\
\sigma(\lambda_\gamma) &= 1/L(\lambda_{\gamma+1} - \lambda_\gamma) \\
\omega(\mu_\beta) &= 1/L(\mu_{\beta+1} - \mu_\beta).
\end{aligned} \tag{11}$$

In terms of those densities, the energy and momentum become

$$E/L = \int k^2 \rho(k)\,\mathrm{d}k \qquad p/L = \int k\rho(k)\,\mathrm{d}k \tag{12}$$

while N, M and M' are determined by

$$N/L = \int \rho(k)\,\mathrm{d}k \qquad M/L = \int \sigma(\lambda)\,\mathrm{d}\lambda \qquad M'/L = \int \omega(\mu)\,\mathrm{d}\mu. \tag{13}$$

As the $SU(3)$ 'magnetic' field is characterized by two parameters H_1 and H_2, the Zeeman term is given by

$$\begin{aligned}\mathcal{H}_{\mathrm{zee}} &= H_1(N - 2M + M')/2 + H_2(M - 2M')/2 \\ &= \frac{H_1 L}{2}\int \rho(k)\,\mathrm{d}k + \frac{(H_2 - 2H_1)L}{2}\int \sigma(\lambda)\,\mathrm{d}\lambda + \frac{(H_1 - 2H_2)L}{2}\int \omega(\mu)\,\mathrm{d}\mu.\end{aligned} \tag{14}$$

3. The ground state

It is easy to show that the first equation of equation (9) is a monotonically increasing function of k_j, i.e. if $I_i < I_j$ we have $k_i < k_j$. So the configuration of $\{I_j\}$ for the ground state is given by successive integers or half-integers symmetrically arranged around zero, i.e. $I_{j+1} - I_j = 1$. In order to observe the properties of $\{J_\gamma, J'_\beta\}$, it is useful to investigate equations (9) in the weak coupling limit $c \to 0$. Due to $\Theta_{\pm n}(x) \to \mp\pi\,\mathrm{sgn}(x)$, equations (9) become

$$\begin{aligned}2I_j &= k_j L/\pi + \sum_{l=1}^{N}\mathrm{sgn}(k_j - k_l) - \sum_{\nu=1}^{M}\mathrm{sgn}(k_j - \lambda_\nu) \\ 2J_\gamma &= \sum_{l=1}^{N}\mathrm{sgn}(\lambda_\gamma - k_l) - \sum_{\nu=1}^{M}\mathrm{sgn}(\lambda_\gamma - \lambda_\nu) + \sum_{\alpha=1}^{M'}\mathrm{sgn}(\lambda_\gamma - \mu_\alpha) \\ 2J'_\beta &= \sum_{\nu=1}^{M}\mathrm{sgn}(\mu_\beta - \lambda_\nu) - \sum_{\alpha=1}^{M'}\mathrm{sgn}(\mu_\beta - \mu_\alpha).\end{aligned} \tag{15}$$

We can choose the subscripts of the rapidities k_j, λ_γ, μ_β in such a way that I_j, J_γ, J'_β are all ranged in increasing order. Then we have

$$\begin{aligned}2(I_{j+1} - I_j - 1) &= \frac{L}{\pi}(k_{j+1} - k_j) - \sum_{\nu=1}^{M}[\mathrm{sgn}(k_{j+1} - \lambda_\nu) - \mathrm{sgn}(k_j - \lambda_\nu)] \\ 2(J_{\gamma+1} - J_\gamma + 1) &= \sum_{l=1}^{N}[\mathrm{sgn}(\lambda_{\gamma+1} - k_l) - \mathrm{sgn}(\lambda_\gamma - k_l)] \\ &\quad + \sum_{\alpha=1}^{M'}[\mathrm{sgn}(\lambda_{\gamma+1} - \mu_\alpha) - \mathrm{sgn}(\lambda_\gamma - \mu_\alpha)] \\ 2(J'_{\beta+1} - J'_\beta + 1) &= \sum_{\nu=1}^{M}[\mathrm{sgn}(\mu_{\beta+1} - \lambda_\nu) - \mathrm{sgn}(\mu_\beta - \lambda_\nu)].\end{aligned} \tag{16}$$

Therefore, if $J'_{\beta+1} - J'_\beta = m$, there must exist $m + 1$ solutions of λ_ν satisfying $\mu_\beta < \lambda_\nu < \mu_{\beta+1}$; and if $J_{\gamma+1} - J_\gamma = n$, there must be $n + 1$ solutions of k_l and μ_α satisfying $\lambda_\gamma < k_l$, $\mu_\alpha < \lambda_{\gamma+1}$. So the existence of a λ_ν between two μ has a positive contribution to the density of μ (11), and vice versa for μ to λ. However, from the first equation of equation (16),

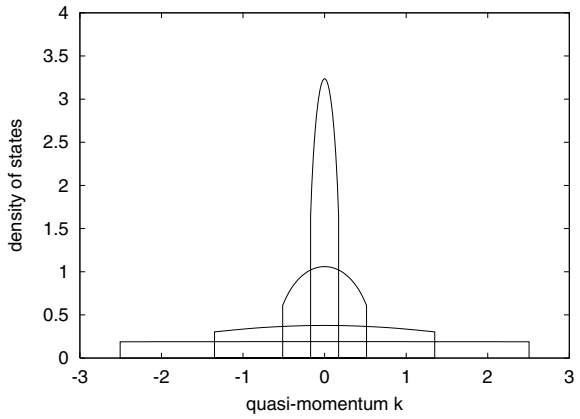

Figure 1. The density of state in k-space for the ground state. The distribution changes gradually from a histogram to a narrow peak for the coupling from strong to weak. The figure is plotted for $N = L = 41$ and $c = 10, 1, 0.1, 0.01$.

for $I_{j+1} - I_j = n$, there will be $k_{j+1} - k_j = 2n\pi/L$ if there is λ_γ such that $k_j < \lambda_\gamma < k_{j+1}$, otherwise $k_{j+1} - k_j = 2(n-1)\pi/L$. So a rapidity of λ_γ always repels the k rapidity away from that value. As a result, an existing λ_γ will suppress the density of state in k-space at the point $k = \lambda_\gamma$. The weaker the coupling, the more magnificent the effect will be. Also, for a given set $\{I\}$, the more λ rapidities there are, the higher the energy is.

It is also useful to observe equation (16) in the strong coupling limit. We consider two cases: $M = 0$ and $M = 1$. For $M = 0$, the secular equation becomes

$$k_j L = 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l) \tag{17}$$

and for $M = 1$ we have

$$k'_j L = 2\pi I'_j + \sum_{l=1}^{N} \Theta_1(k'_j - k'_l) + \Theta_{-1/2}(k'_j - \lambda_1). \tag{18}$$

Here $I_j - I'_j = 1/2$ due to M changing from zero to one. As $c \to \infty$, we have $\tan^{-1}(x/c) \sim x/c$. So the above two equations become

$$(k_{j+1} - k_j)L\left[1 + \frac{2N}{Lc}\right] = 2\pi \qquad (k'_{j+1} - k'_j)L\left[1 + \frac{2(N-1)}{Lc}\right] = 2\pi \tag{19}$$

whence the distribution is almost a histogram. Referring to equation (11) the value of the density distribution for $M = 0$ is larger than that for $M = 1$, which makes the Fermi momentum for the latter case larger than that of the former case so as to keep the total number of particles the same. Therefore, the state of $M = 0$ has lower energy.

Differing from the $SU(3)$ fermionic model [8] and a toy model of the quark cluster [21], the ground state of the $SU(3)$ bosonic model is no longer a colour singlet but a colour ferromagnetic state. The difference is due to the distinct permutation symmetries. For N particles, the ground state is characterized by a one-row N-column Young tableau $[N]$ whose quantum-number configurations are

$$\left\{I_j^0\right\} = \{-(N-1)/2, \ldots, (N-1)/2\} \qquad M = M' = 0. \tag{20}$$

The density of states for the ground state is plotted in figure 1 for various couplings with $L = N = 41$.

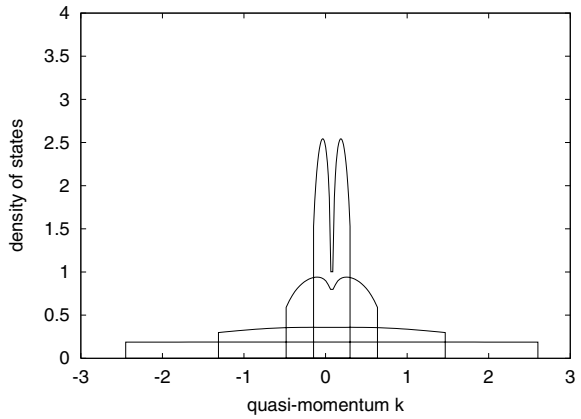

Figure 2. The density of state in k-space for the ground state in the presence of one colour rapidity by choosing $J_1 = 0$. The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $N = L = 100$ and $c = 10, 1, 0.1, 0.01$.

In the thermodynamic limit, the density corresponding to the configuration of quantum numbers of the ground state satisfies the integral equation

$$\rho_0(k) = \frac{1}{2\pi} + \int_{-k_F}^{k_F} K_2(k - k')\rho_0(k')\,\mathrm{d}k'. \tag{21}$$

Here $\rho_0(k)$ and k_F are the density and integration limit for the ground state, respectively, and

$$K_n(x) = \frac{1}{\pi}\frac{nc/2}{n^2c^2/4 + x^2}.$$

The concentration is given by

$$D = N/L = \int_{-k_F}^{k_F} \rho_0(k)\,\mathrm{d}k. \tag{22}$$

From equations (21) and (22), we can determine $\rho_0(k)$ and k_F. Here k_F is a quasi-Fermi momentum because the wavefunction vanishes for any $k_j = k_l$ $(j \neq l)$ as long as $c \neq 0$ even in the Bose system, which can be seen from equation (7). The energy can be calculated by

$$E_0/L = \int_{-k_F}^{k_F} k^2\rho_0(k)\,\mathrm{d}k \tag{23}$$

which is explicitly $\frac{1}{3}\pi^2 D^3\left(1 - \frac{4}{c}D\right)$ in the strong coupling limit. In the general case, we need to solve the equations numerically (figure 2). We show the ground-state energy for particle densities $D = 1.0, 0.75, 0.5$ in figure 3.

4. Low-lying excited states

The low-lying excited states are obtained by varying the configuration $\{I_j, J_\gamma, J'_\beta\}$ from that of the ground state.

4.1. Holon–antiholon excitation

The simplest case is to remove one of I from the configuration of the ground state and add a new one outside the original sequence, i.e.,

$$\{I_j\} = \{-(N-1)/2, \ldots, n_1 - 1, n_1 + 1, \ldots, (N-1)/2, I_n\}$$

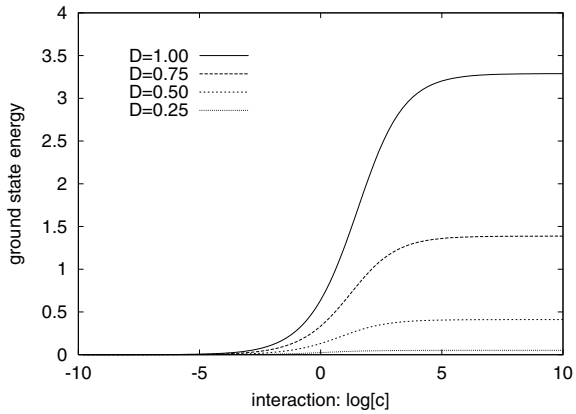

Figure 3. The ground-state energy E/L versus the coupling constant $\ln c$ for different densities $D = 1.0, 0.75, 0.5, 0.25$.

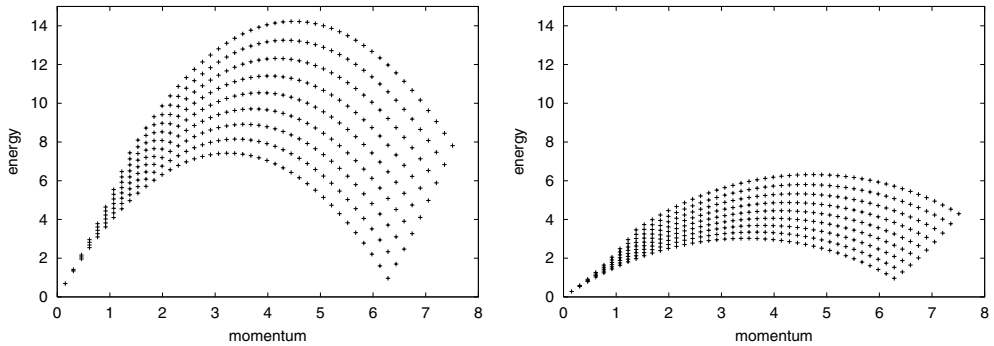

Figure 4. The holon–antiholon excitation spectrum calculated for $N = L = 41$, and $c = 10$ (left) and $c = 1$ (right).

where $|I_n| > (N-1)/2$ and $M = M' = 0$. We call this the holon–antiholon excitation, which consists of a 'holon' created under the Fermi surface and an 'antiholon' created outside it. In figure 4, we plot the numerical results of the energy-momentum spectrum for a system with $L = N = 41$ (the other part is just the mirror image of the plotted part corresponding to the state with $p \to -p$ coming from the negative I_n). From the figure, we notice that there is a minimum in the excitation energy at $p = 2\pi$ due to the fact that both I_1^0 replaced by $I_n = (N+1)/2$ and I_N^0 share the same energy; their momenta difference, however, is 2π. The overall structure of the spectrum is not changed obviously between $c = 1$ and $c = 10$. For a system of finite size, the gap of holon–antiholon excitations opens. In the thermodynamic limit, however, it vanishes.

In the configuration of quantum numbers for the ground state (20), replacing $I_N^0 = (N-1)/2$ by $I_N^0 = (N-1)/2 + n, n = 1, 2, \ldots$ and keeping the others unchanged, we obtain the dispersion relation of antiholon (figure 5) by solving the Bethe ansatz equations (9) numerically. In a similar way, replacing $I_n^0, n = 1, \ldots, N$ of $\{I_j^0\}$ in turn by $(N+1)/2$, we have the dispersion relation of the holon, as shown in figure 6.

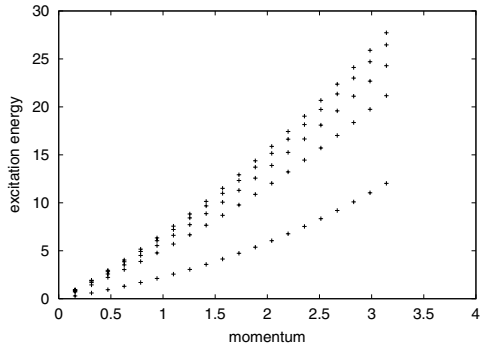

Figure 5. The dispersion relation of antiholon excitation for different coupling constants where the curves from bottom to top correspond to $c = 1$, 10, 20, 40 and 80, respectively. Here $N = L = 40$.

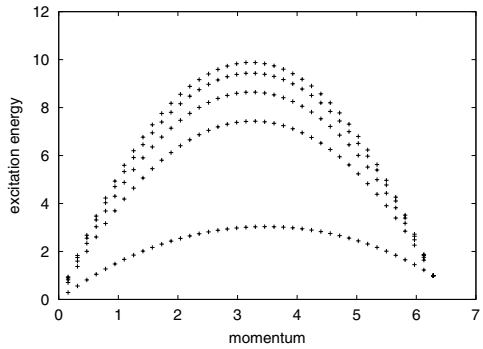

Figure 6. The dispersion relation of the holon for different coupling constants where the curves from bottom to top correspond to $c = 1$, 10, 20, 40 and 80, respectively. Here $N = L = 40$.

In the thermodynamic limit, it is plausible to calculate the excitation energy by making $\rho(k) = \rho_0(k) + \rho_1(k)/L$ where $\rho_0(k)$ is the density of the ground state. By creating a hole inside the quasi-Fermi sea $\bar{k} \in [-k_F, k_F]$ and an additional $k_p > k_F$ outside it, we have

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} \mathrm{d}k'\, \rho_1(k') K_2(k - k') + K_2(k - k_p). \tag{24}$$

The excitation energy consists of two terms $\triangle E = \int k^2 \rho_1(k)\,\mathrm{d}k + k_p^2 = \varepsilon_h(\bar{k}) + \varepsilon_a(k_p)$. The holon energy ε_h and antiholon energy $\varepsilon_a(k_p) = -\varepsilon_h(k_p)$ are given by

$$\begin{aligned} \varepsilon_h(\bar{k}) &= -\bar{k}^2 + \int_{-k_F}^{k_F} k^2 \rho_1^h(k, \bar{k})\,\mathrm{d}k \\ \rho_1^h(k, \bar{k}) &= -K_2(k - \bar{k}) + \int_{-k_F}^{k_F} K_2(k - k')\rho_1^h(k', \bar{k})\,\mathrm{d}k'. \end{aligned} \tag{25}$$

4.2. Holon–coloron excitation

Excitations related to the colour sector are characterized by adding λ and μ rapidities into the system. The simplest excitation of this type is obtained by considering $M = 1$, which is labelled $(N/2 - 1, 1/2)$. Compared to the ground state, the quantum number changes from

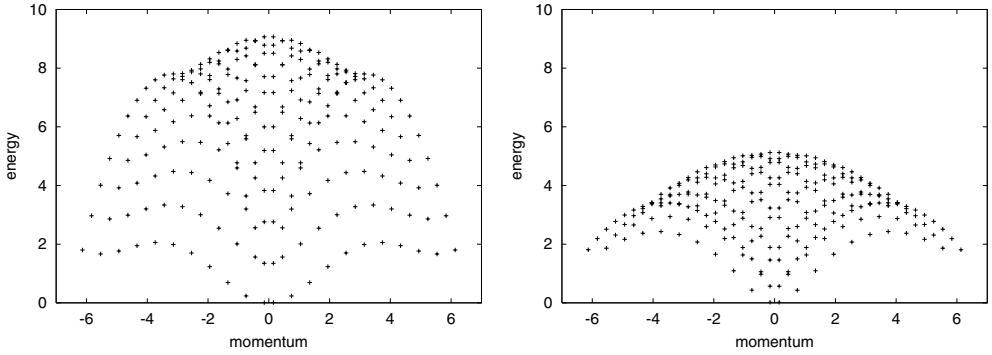

Figure 7. The holon–coloron(σ type) excitation spectrum calculated for $N = L = 21$, and $c = 10$ (left) and $c = 1$ (right).

half-integer to integer, or vice versa. We call this type of excitation, σ-coloron, which is regarded as an elementary quasi-particle of the present model. Its quantum number takes

$$I_1 = -N/2 + \delta_{1,j_1} \quad (1 \leqslant j_1 \leqslant N+1)$$
$$I_j = I_{j-1} + 1 + \delta_{j,j_1} \quad (j = 2, \ldots, N)$$

while $J_1 = I_1 + m$ ($m = 1, 2, \ldots, N-1$) so that $I_1 < J_1 < I_N$. This produces $N-1$ multiplets. The excitation spectra are plotted in figure 7 for a system of $N = L = 21$ with $c = 10, 1$, respectively.

Adding an additional λ rapidity to the colour ferromagnetic ground state brings about a hole in the k-sector. Now we have two-parameter excitation $\triangle E = \int k^2 \rho_1(k)\,\mathrm{d}k$ where the $\rho_1(k)$ solves

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} K_2(k-k')\rho_1(k')\,\mathrm{d}k' - K_1(k-\lambda). \tag{26}$$

The energy of the holon–coloron excitation consists of two terms $\triangle E = \varepsilon_h(\bar{k}) + \varepsilon_c(\lambda)$. ε_h is determined by equations (25) and ε_c is defined by $\varepsilon_c(\lambda) = \int k^2 \rho_1^c(k,\lambda)$ with

$$\rho_1^c(k,\lambda) = -K_1(k-\lambda) + \int_{-k_F}^{k_F} K_2(k-k')\rho_1^c(k',\lambda)\,\mathrm{d}k'. \tag{27}$$

$\varepsilon_h(\bar{k})$ and $\varepsilon_c(\lambda)$ are energies of the holon and σ-coloron whose dispersions are shown in figures 6 and 9, respectively.

Furthermore, the overall structure for the case of $c = 1$ in figure 7 differs from the case of $c = 10$. We interpret the phenomenon as being due to the fact that the dependence of the dispersion relations of the holon and σ-coloron on the coupling constant are different. This feature can be concluded from figures 6 and 9. When c decreases, $\varepsilon_h(p)$ decreases while $\varepsilon_c(p)$ increases.

4.3. The σ-type coloron–coloron excitation

Leaving the configuration of the ground state $\{I_j^0\}$ unchanged and changing M from zero to $M = 2$, which corresponds to $(N/2-2, 1)$, a two-parameter excitation in the λ-sector is characterized by

$$-(N-1)/2 < J_1 < J_2 < (N-1)/2.$$

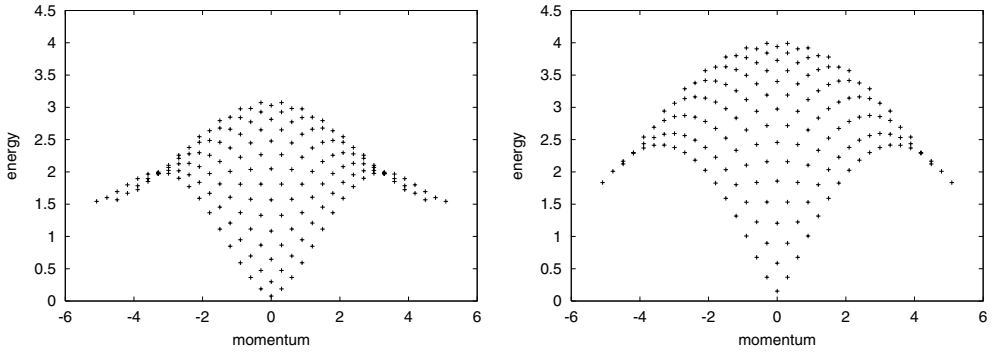

Figure 8. The coloron(σ type)–coloron(σ type) excitation spectrum calculated for $N = L = 21$, and $c = 10$ (left) and $c = 1$ (right).

There are in total $N(N-3)/2$ possible choices for such a type of excitation. In figure 8, we plot the numerical result of the energy-momentum spectrum for a system with $N = L = 21$.

The excitation energy of coloron(σ type)–coloron (σ type) can be calculated by $\triangle E = \int k^2 \rho_1^c(k, \lambda_1, \lambda_2)\,\mathrm{d}k$, where $\rho_1^c(k, \lambda_1, \lambda_2)$ is determined by

$$\rho_1^c(k, \lambda_1, \lambda_2) = -K_1(k - \lambda_1) - K_1(k - \lambda_2) + \int_{-k_F}^{k_F} K_2(k - k')\rho_1^c(k', \lambda_1, \lambda_2)\,\mathrm{d}k'. \tag{28}$$

It consists of two terms $\triangle E = \varepsilon_c(\lambda_1) + \varepsilon_c(\lambda_2)$, where the coloron energy $\varepsilon_c(\lambda)$ has been give in the text before equation (27).

4.4. Dispersion relation of the ω-coloron

The fourth possible excitation involves both the additional quantum numbers J and J'. For $M = 2$ and $M' = 1$ there is no range for J' varying, but for large M the excitation is no longer low-lying excitation. So we only show its dispersion relation, which is described by the following configuration

$$\begin{aligned} &\{I_j\} = \{-(N-1)/2, \ldots, (N-1)/2\} \\ &\{J_\gamma\} = \{-M/2, \ldots, (M-2)/2\} \\ &J'_1 = -M/2 + 1, \ldots, M/2 - 1 \end{aligned} \tag{29}$$

for a given M. We have plotted the dispersion relation of the ω-coloron in figure 10 by varying J'_1 for a system with $L = N = 40$. The figure has a minimum around $p = \pi$ when $M = N/2$.

Up to now, we have discussed three low-lying excitation energies and the dispersion relations of four possible elementary particles: holon, antiholon, σ-coloron and ω-coloron. We have found that these low-lying excitations are gapless in the thermodynamic limit (figures 5, 6, 9, 10).

5. Thermodynamics at finite temperature

For the ground state (i.e. at zero temperature), the charge rapidities k are the real roots of the Bethe ansatz equations (9). For the excited state, however, the λ and μ rapidities can be complex roots [16, 17] which are always from a 'bound state' with several other λ. This arises from the consistency of both sides of the Bethe ansatz equations [18] in the limit $L \to \infty$,

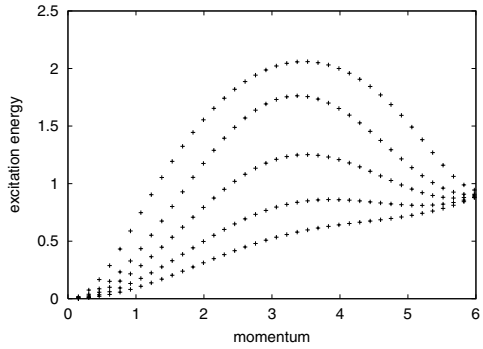

Figure 9. The dispersion relation of σ-coloron for different coupling constants where the curves from top to bottom correspond to $c = 1$, 10, 20, 40 and 80, respectively. Here $N = L = 41$.

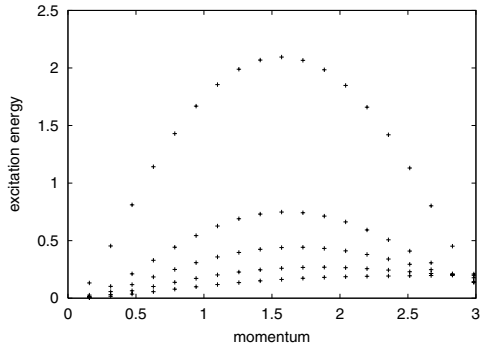

Figure 10. The dispersion relation of ω-coloron for different coupling constants where the curves from top to bottom correspond to $c = 1$, 10, 20, 40 and 80, respectively. Here $N = L = 40$. Zero energy corresponds to the $M = N/2$ ground state.

$N \to \infty$. The n-string rapidity is defined by

$$\begin{aligned} \Lambda_a^{nj} &= \lambda_a^n + (n+1-2j)iu + O(\exp(-\delta N)) \\ U_a^{nj} &= \mu_a^n + (n+1-2j)iu + O(\exp(-\delta N)) \end{aligned} \tag{30}$$

where $u = c/2$, $j = 1, 2, \ldots, n$. The total numbers of λ and μ are determined by

$$M = \sum_{n=1}^{\infty} nM_n \qquad M' = \sum_{n=1}^{\infty} nM_n' \tag{31}$$

where M_n and M_n' denote the number of λ n-strings and μ n-strings, respectively. Equations (9) become

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_l \Theta_1(k_j - k_l) + \sum_{an} \Theta_{-n/2}\left(k_j - \lambda_a^n\right) \\ 2\pi J_a^n &= \sum_l \Theta_{-n/2}\left(\lambda_a^n - k_l\right) + \sum_{bl,t\neq 0} A_{nlt}\Theta_{t/2}\left(\lambda_a^n - \lambda_b^l\right) + \sum_{clt} B_{nlt}\Theta_{-t/2}\left(\lambda_a^n - \mu_c^l\right) \\ 2\pi J'^n_a &= \sum_{blt} B_{nlt}\Theta_{t/2}\left(\mu_a^n - \lambda_b^l\right) + \sum_{cl,t\neq 0} A_{nlt}\Theta_{-t/2}\left(\mu_a^n - \mu_c^l\right) \end{aligned} \tag{32}$$

where

$$A_{nlt} = \begin{cases} 1 & \text{for} \quad t = n + l, |n - l| \\ 2 & \text{for} \quad t = n + l - 2, \ldots, |n - l| + 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_{nlt} = \begin{cases} 1 & \text{for} \quad t = n + l - 1, n + l - 3, \ldots, |n - l| + 1 \\ 0 & \text{otherwise} \end{cases}$$

and the quantum numbers $\{I_j, J_a^n, J_a'^n\}$ label the state which is no longer the ground state. Replacing $k_j, \lambda_a^n, \mu_a^n$ in equations (32) by continuous variables k, λ, μ but still keeping the summation over the solutions of these roots, we can consider the quantum numbers $I_j, J_a^n, J_a'^n$ as functions $I(k)$, $J^n(\lambda)$ and $J'^n(\mu)$ given by equations (32). Taking $I(\lambda)$ as an example, when $I(k)$ passes through one of the quantum numbers I_j, the corresponding k is equal to one of the roots k_j, as for $J^n(\lambda)$ and $J'^n(\mu)$. However, there may exist some integers or half-integers for which the corresponding $k(\lambda, \mu)$ is not in the set of roots. Such a situation is conventionally referred to as a 'hole'. In the thermodynamic limit, we may introduce the densities of real k, λ n-string and μ n-string

$$\begin{aligned} &\rho(k) + \rho^h(k) = (1/L)\,\mathrm{d}I(k)/\mathrm{d}k \\ &\sigma_n(\lambda) + \sigma_n^h(\lambda) = (1/L)\,\mathrm{d}J^n(\lambda)/\mathrm{d}\lambda \\ &\omega_n(\mu) + \omega_n^h(\mu) = (1/L)\,\mathrm{d}J'^n(\mu)/\mathrm{d}\mu. \end{aligned} \tag{33}$$

Then equations (32) give rise to the following coupled integral equations:

$$\begin{aligned} \rho + \rho^h &= \frac{1}{2\pi} + \int K_2(k - k')\rho(k')\,\mathrm{d}k' - \sum_n \int K_n(k - \lambda)\sigma_n(\lambda)\,\mathrm{d}\lambda \\ \sigma_n^h &= \int K_n(\lambda - k)\rho(k)\,\mathrm{d}k - \sum_{lt} A_{nlt} \int K_t(\lambda - \lambda')\sigma_l(\lambda')\,\mathrm{d}\lambda' \\ &\quad + \sum_{lt} B_{nlt} \int K_t(\lambda - \mu)\omega_l(\mu)\,\mathrm{d}\mu \\ \omega_n^h &= \sum_{lt} B_{nlt} \int K_t(\mu - \lambda)\sigma_l(\lambda)\,\mathrm{d}\lambda - \sum_{lt} A_{nlt} \int K_t(\mu - \mu')\omega_l(\mu')\,\mathrm{d}\mu'. \end{aligned} \tag{34}$$

σ_n and ω_n arising from the definition (34) that occur in the left-hand side have been moved to the right-hand side by including the $t = 0$ term in the summation. In terms of densities defined above, the total numbers of λ and μ are given by

$$M/L = \sum_n n \int \sigma_n(\lambda)\,\mathrm{d}\lambda \qquad M'/L = \sum_n n \int \omega_n(\mu)\,\mathrm{d}\mu. \tag{35}$$

In the presence of the $SU(3)$ magnetic fields H_1 and H_2, we can define two types of 'magnetization' whose z-components are

$$\begin{aligned} T^z/L &= \frac{1}{2}\int \rho(k)\,\mathrm{d}k - \sum_n n \int \sigma_n(\lambda)\,\mathrm{d}\lambda + \frac{1}{2}\sum_n n \int \omega_n(\mu)\,\mathrm{d}\mu \\ U^z/L &= \frac{1}{2}\sum_n n \int \sigma_n(\lambda)\,\mathrm{d}\lambda - \sum_n n \int \omega_n(\mu)\,\mathrm{d}\mu. \end{aligned} \tag{36}$$

Hence the energy contributed by the Zeeman term (14) is

$$E_{\mathrm{Zee}} = H_1 T^z + H_2 U^z. \tag{37}$$

For given $\rho(k)$, $\rho^h(k)$, $\sigma_n(\lambda)$, $\sigma_n^h(\lambda)$, $\omega_n(\mu)$ and $\omega_n^h(\mu)$ the entropy has the form [19]

$$\begin{aligned}\mathcal{S}/L = &\int [(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h\ln\rho^h]\,\mathrm{d}k \\ &+ \sum_n \int \left[\left(\sigma_n+\sigma_n^h\right)\ln\left(\sigma_n+\sigma_n^h\right) - \sigma_n\ln\sigma_n - \sigma_n^h\ln\sigma_n^h\right]\mathrm{d}\lambda \\ &+ \sum_n \int \left[\left(\omega_n+\omega_n^h\right)\ln\left(\omega_n+\omega_n^h\right) - \omega_n\ln\omega_n - \omega_n^h\ln\omega_n^h\right]\mathrm{d}\mu. \end{aligned} \tag{38}$$

where the Boltzmann constant is set to unity.

At finite temperature, the thermal equilibrium is obtained by minimizing the free energy $F = E - E_{\mathrm{Zee}} - T\mathcal{S} - \mu N$ where μ is the chemical potential and $\mathcal{S}$ is the entropy of the system. Making use of the relations derived from equations (34)

$$\begin{aligned}\delta\rho^h &= -\delta\rho + \int K_2(k-k')\delta\rho\,\mathrm{d}k' - \sum_n \int K_n(k-\lambda)\delta\sigma_n\,\mathrm{d}\lambda \\ \delta\sigma_n^h &= \int K_n(\lambda-k)\delta\rho\,\mathrm{d}k - \sum_{lt} A_{nlt}\int K_t(\lambda-\lambda')\delta\sigma_l(\lambda')\,\mathrm{d}\lambda' \\ &\quad + \sum_{lt} B_{nlt}\int K_t(\lambda-\mu)\delta\omega_l(\mu)\,\mathrm{d}\mu \\ \delta\omega_n^h &= \sum_{lt} B_{nlt}\int K_t(\mu-\lambda)\delta\sigma_l(\lambda)\,\mathrm{d}\lambda - \sum_{lt} A_{nlt}\int K_t(\mu-\mu')\delta\omega_l(\mu')\,\mathrm{d}\mu' \end{aligned} \tag{39}$$

and we define

$$\frac{\rho^h(k)}{\rho(k)} = \kappa(k) = \mathrm{e}^{\epsilon(k)/T} \qquad \frac{\sigma_n^h(\lambda)}{\sigma_n(\lambda)} = \eta_n(\lambda) = \mathrm{e}^{\zeta_n(\lambda)/T} \qquad \frac{\omega_n^h(\mu)}{\omega_n(\mu)} = \Delta_n(\mu) = \mathrm{e}^{\xi_n(\mu)/T}. \tag{40}$$

We obtain the following conditions from the minimum condition $\delta F = 0$, namely

$$\begin{aligned}\epsilon(k) &= k^2 - \mu - H_1/2 - T\int K_2(k-k')\ln(1+\mathrm{e}^{-\epsilon(k')/T})\,\mathrm{d}k' \\ &\quad - T\sum_n \int K_n(k-\lambda)\ln[1+\mathrm{e}^{-\zeta_n(\lambda)/T}]\,\mathrm{d}\lambda \\ \zeta_n(\lambda) &= n(2H_1-H_2)/2 + T\int K_n(\lambda-k)\ln[1+\mathrm{e}^{-\epsilon(k)/T}]\,\mathrm{d}k \\ &\quad + T\sum_{l,t\neq 0} A_{nlt}\int K_t(\lambda-\lambda')\ln[1+\mathrm{e}^{-\zeta_l(\lambda')/T}]\,\mathrm{d}\lambda' \\ &\quad - T\sum_{lt} B_{nlt}\int K_t(\lambda-\mu)\ln[1+\mathrm{e}^{-\xi_l(\mu)}]\,\mathrm{d}\mu \\ \xi_n(\mu) &= n(2H_2-H_1)/2 - T\sum_{lt} B_{nlt}\int K_t(\mu-\lambda)\ln[1+\mathrm{e}^{-\zeta_l(\lambda)}]\,\mathrm{d}\lambda \\ &\quad + T\sum_{l,t\neq 0} A_{nlt}\int K_t(\mu-\mu')\ln[1+\mathrm{e}^{-\xi_l(\mu')/T}]\,\mathrm{d}\mu'. \end{aligned} \tag{41}$$

A more useful version of equations (41) is the recursive scheme which is a revised version of the Gaudin–Takahashi equations, as obtained by Fourier transformation:

$$
\begin{aligned}
T\ln\kappa &= k^2-\mu-H_1/2-TK_2(k)*\ln[1+\kappa^{-1}]-T\sum_n K_n(k)*\ln\left[1+\eta_n^{-1}\right]\\
\ln\eta_1 &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u)*\ln\left[(1+\kappa^{-1})(1+\eta_2)\Big/\left(1+\Delta_1^{-1}\right)\right]\\
\ln\eta_n &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u)*\ln\left[(1+\eta_{n-1})(1+\eta_{n+1})\Big/\left(1+\Delta_n^{-1}\right)\right]\\
\ln\Delta_1 &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u)*\ln\left[(1+\Delta_2)\Big/\left(1+\eta_1^{-1}\right)\right]\\
\ln\Delta_n &= \frac{1}{4u}\mathrm{sech}(\pi\lambda/2u)*\ln\left[(1+\Delta_{n-1})(1+\Delta_{n+1})\Big/\left(1+\eta_n^{-1}\right)\right].
\end{aligned}
\tag{42}
$$

Here * denotes a convolution. Also, these equations are completed by the asymptotic conditions

$$
\lim_{n\to\infty}[\ln\eta_n/n]=(2H_1-H_2)/2T \qquad \lim_{n\to\infty}[\ln\Delta_n/n]=(2H_2-H_1)/2T. \tag{43}
$$

Finally, we obtain the Helmholtz free energy $F=E-T\mathcal{S}$

$$
F=\mu N-\frac{LT}{2\pi}\int\ln[1+\mathrm{e}^{-\epsilon}]\,\mathrm{d}k \tag{44}
$$

and the pressure of the system

$$
P=-\frac{\partial F}{\partial L}=\frac{T}{2\pi}\int\ln[1+\mathrm{e}^{-\epsilon}]\,\mathrm{d}k \tag{45}
$$

which is formally the same as the expression of Yang and Yang [19] but the equation which ϵ fulfils is different.

6. Special cases

In general, the free energy can be calculated by using formula (44), where $\epsilon(k)$ and $\zeta_n(\lambda)$ are determined from equations (41) which can be solved by iteration. In the following, we consider some special cases because explicit results are obtainable in those cases.

6.1. Zero-temperature limit

The state at zero temperature is the ground state. When $T\to 0$, the first equation of equation (41) becomes

$$
\epsilon(k)=k^2-\mu-H_1/2+\int K_2(k-k')\epsilon(k')\,\mathrm{d}k'+\sum_n\int K_n(k-\lambda)\zeta_n(\lambda)\,\mathrm{d}\lambda. \tag{46}
$$

Then the Fermi surface is determined by $\epsilon(k_F)=0$. Since there is no hole under the Fermi surface, we can take the ratio $\kappa=\rho^h/\rho$ as zero when $k\in[-k_F,k_F]$. As a result, it is easy to see from equations (42) that $\eta_n=\Delta_n\to\infty$. That is, $M=M'=0$, and the state is a colour ferromagnetic state. This is consistent with the conclusion obtained in section 4. Then equation (46) can be rewritten as

$$
\epsilon_0(k)=k^2-\mu-H_1/2+\int_{-k_F}^{k_F}K_2(k-k')\epsilon_0(k')\,\mathrm{d}k' \tag{47}
$$

which gives the solution of dressed energy [20], and the ground-state energy can be given in terms of ϵ_0

$$E_0/L = \frac{1}{2\pi}\int_{-k_F}^{k_F} \epsilon_0(k)\,\mathrm{d}k \tag{48}$$

whose dependence on the coupling constant is shown in figure 10.

6.2. High-temperature limit

In the high-temperature limit $T \to \infty$, we can assume that all functions $\eta_n(\lambda)$ and $\Delta_n(\mu)$ are independent of their corresponding parameter. Due to $\lim_{u\to 0}\frac{1}{2u}\mathrm{sech}\left(\frac{\pi\lambda}{2u}\right) = \delta(\lambda)$, equations (42) can be written as follows

$$\begin{aligned} &\eta_1^2 = (1+\kappa^{-1})(1+\eta_2)\big/\left(1+\Delta_1^{-1}\right) \qquad && \eta_n^2 = (1+\eta_{n-1})(1+\eta_{n+1})\big/\left(1+\Delta_n^{-1}\right) \\ &\Delta_1^2 = (1+\Delta_2)\big/\left(1+\eta_1^{-1}\right) && \Delta_n^2 = (1+\Delta_{n-1})(1+\Delta_{n+1})\big/\left(1+\eta_n^{-1}\right) \end{aligned} \tag{49}$$

with the asymptotic conditions (43).

Performing the Fourier transform to equations (34), we obtain the solution of the densities of λ n-strings:

$$\begin{aligned} \sigma_1 + \sigma_1^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * \left[\rho + \sigma_2^h + \omega_1\right] \\ \sigma_n + \sigma_n^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * \left[\sigma_{n-1}^h + \sigma_{n+1}^h + \omega_n\right] \\ \omega_1 + \omega_1^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * \left[\omega_2^h + \sigma_1\right] \\ \omega_n + \omega_n^h &= \frac{1}{4u}\mathrm{sech}[\pi\lambda/2u] * \left[\omega_{n-1}^h + \omega_{n+1}^h + \sigma_n\right]. \end{aligned} \tag{50}$$

If we assume that σ_n, σ_n^h and ω_n, ω_n^h are independent of λ and μ, respectively, or letting $u = 0$, we have the following relation

$$\begin{aligned} \sum_n n\sigma_n &= \frac{\rho}{2} + \frac{1}{2}\sum_n n\omega_n - \frac{n_m+1}{2}\sigma_{n_m}\,\mathrm{e}^{n_m\Omega_1/T} \\ \sum_n n\omega_n &= \frac{1}{2}\sum_n n\sigma_n - \frac{n_{m'}+1}{2}\sigma_{n_{m'}}\,\mathrm{e}^{n_{m'}\Omega_2/T} \end{aligned} \tag{51}$$

where n_m and $n_{m'}$ are the maximal length of the λ string and μ string, respectively, and $\Omega_1 = 2H_1 - H_2$, $\Omega_2 = 2H_2 - H_1$. In the absence of external fields H_1 and H_2, it is easy to obtain $M' = M/2 = N/3$ which means there are $N/3$ particles in each internal state. Then we can also infer that the contribution of the internal degree of freedom to the entropy per site must be $\mathcal{S} = \ln 3$, following from the fact that the internal degree of freedom per particle is three.

If the external field is small, expanding equation (51) for the small field and integrating the equation over λ and μ, we obtain the $SU(3)$ magnetization of the model

$$\begin{aligned} \frac{T^z}{L} &= \frac{M_m}{2L}\left[1 + \frac{n_m\Omega_1}{T} + \frac{1}{2}\left(\frac{n_m\Omega_1}{T}\right)^2 + \cdots\right] \\ \frac{U^z}{L} &= \frac{M'_m}{2L}\left[1 + \frac{n_{m'}\Omega_2}{T} + \frac{1}{2}\left(\frac{n_{m'}\Omega_2}{T}\right)^2 + \cdots\right] \end{aligned} \tag{52}$$

where M_m and $M_{m'}$ are the total number of rapidities in λ n_m-strings and μ $n_{m'}$-strings, respectively. The first term in the parentheses of both equations arises from self-magnetization,

while the others are contributed by the external field. Equation (52) indicates that the magnetic property of the model in the high-temperature regime is dominated by Curie's law, $\chi \propto 1/T$.

6.3. The strong coupling limit

For $u \to \infty$, $K_n(k)$ goes to zero, and from equations (41) we have

$$\epsilon = k^2 - \mu \tag{53}$$

where the external field is set to unity. The k-sector is completely decoupled with λ- and μ-sectors. At arbitrary temperature, the solutions for η_n and Δ_n are independent of parameters λ and μ, respectively, which gives rise to equation (49). The free energy of the system defined by equation (44) can be solved by integration by part

$$F/L = \mu D - \frac{2}{\pi}\left[\frac{1}{3}\mu^{3/2} + \frac{T^2\pi^2}{24\mu^{1/2}}\right] \tag{54}$$

where the external field is set to zero.

We are not able to deduce the specific heat directly from the free energy obtained above because the chemical potential is a function of temperature. From equations (34), the density of charge rapidity has the form

$$\rho = \frac{1}{2\pi}\frac{1}{1+\mathrm{e}^{(k^2-\mu)/T}}. \tag{55}$$

Integrating the charge density over k space with the condition (13), we have an explicit expression of the chemical potential

$$\mu = \mu_0\left[1 - \frac{\pi^2T^2}{24\mu_0^2}\right]^{-2} \tag{56}$$

where $\mu_0 = \pi^2D^2$, which denotes μ at zero temperature. Then the free energy becomes

$$F/L = \mu_0 D\left[1 + \frac{\pi^2T^2}{12\mu_0^2}\right] - \frac{2}{3\pi}\mu_0^{3/2}\left[1 + \frac{\pi^2T^2}{4\mu_0^2}\right]. \tag{57}$$

The free energy for the $SU(3)$ invariant spin chain also has a T^2 dependence [22].

Since in thermodynamics $S = -\partial F/\partial T$ and $C_v = T\partial S/\partial T$, we find the specific heat at low temperature is Fermi-liquid-like:

$$S = C_v = \frac{T}{6D}. \tag{58}$$

This is the same as the result of the one-component case, since for the strong coupling limit the colour degree of freedom and the charge degree of freedom are decoupled completely, and the contribution of the colour degree of freedom to the free energy vanishes.

7. Conclusions

In this paper, we have solved one-dimensional $SU(3)$ bosons with δ-function interaction using the coordinate Bethe ansatz method. On the basis of the Bethe ansatz equations, we first discussed the ground state of the Bose system and found that the ground state is a colour ferromagnetic state which differs greatly from the $SU(3)$ Fermi system. The configuration of quantum numbers for the ground state was given explicitly. The low-lying excitations were discussed extensively by both analytical and numerical methods. The energy-momentum spectra for three types of excitations, holon–antiholon, holon–coloron(σ type) and coloron

(σ type)–coloron(σ type), were plotted for $c = 10$ and $c = 1$. We also discussed the dispersion relations of four elementary quasi-particles.

The thermodynamics of the system were studied using the TBA strategy. A revised version of the Gaudin–Takahashi equations was obtained by minimizing the free energy at finite temperature. We have found that the magnetic property of the system at the high-temperature regime is dominated by Curie's law, and for the case of strong coupling the system possess Fermi-liquid-like properties and its specific heat is a linear function of T at low temperature.

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